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ON MATHEMATICAL MODELLING OF SOME ISSUES ARISING IN THE OPTIMIZATION OF BIOGAS PRODUCTION IN SOLID WASTE LANDFILLS

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Abstract. In this paper, we address certain issues that arise in the management of processes in open landfills of solid consumption waste for optimizing biogas extraction processes. The paper provides a relatively detailed overview of the characteristics of the "Ķīvītes" solid waste landfill (which serves as the technological base for LLC "Liepājas RAS"), located in the open air in the Grobiņa district of the Kurzeme region in Latvia. This landfill was the site of a major scientific and innovative project (2010/0301/2DP/2.1.1.1.0/10/APIA/VIAA/151) implemented successfully by the authors of this paper between 2011 and 2013. This project, titled "New technology and software development for biogas extraction processes optimization," received financial support from the European Regional Development Fund (ERDF) and the Liepaja city council, Latvia. The paper demonstrates one of the possible approaches – the application of mathematical modelling – to addressing the issues that arise in the context of landfill biogas production optimization.

Keywords: *Solid waste landfill, biogas production, mathematical modelling, direct problem, inverse problem.*

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1. Introduction

The human, by generating a scientific idea within the social environment, has introduced a new geological force into the biosphere, previously non-existent in it (Vernadsky, 1991). The biosphere has transitioned into a new evolutionary state, being reshaped by the scientific thoughts of human society. Through its impact on nature, human society has transformed into the greatest geological force, approaching the scale of raw materials extraction and processing comparable to the activity of the biota within the biosphere and has already surpassed volcanic activity. Currently, each of the three dimensions – Earth's population, industrial production volume and industrial waste – follows an exponential growth law over time. The only difference lies in the value of the coefficient determining the steepness of this growth. Meanwhile, environmental pollution from industrial waste, domestic waste, garbage and offal is increasing at a faster rate than the global population (Borovsky, 2001). Such an implacable increase in production and consumption waste – the "waste crisis" – is one of the spot-on environmental issues of the modern world. From the perspective of industrial ecology, production and consumption waste refers to waste in a solid state, as well as certain gaseous and liquid wastes that can transition into a solid phase (i.e. in filters or sedimentation tanks). The main types of solid waste can be divided into two classes (see Figure 1): waste products of production and consumption waste.

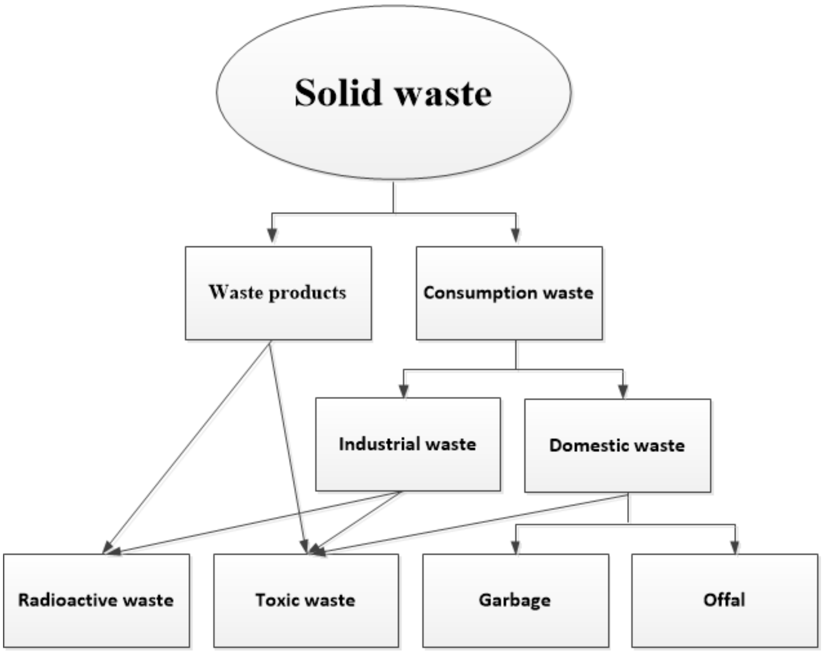

Figure 1. Classification of the main types of municipal solid waste.

Waste products include everything generated during the production process or after its completion, excluding products in the form of energy or material objects of production. According to this definition, production waste includes the residues of multicomponent natural raw materials after extracting the target product, such as empty ore rocks, overburden in mining operations, slag and ash from thermal power plants, furnace slags, burned-earth from metallurgical production and metal shavings from machine-building enterprises and so on. Moreover, waste products include significant waste from the forestry, wood processing, textile and other industries, as well as from the road construction industry and the modern agro-industrial complex (manure storages, unused chemical fertilizers and pesticides, undeveloped burial sites for animals that perished during epidemics, etc.). Substances present in the emitted process gases (combustion/furnace gases) or in the wastewater of enterprises utilizing water in their technological processes are also considered as part of production waste. These gaseous and liquid types of waste are typically addressed in the frames of environmental issues related to air and water pollution of the Earth's atmosphere and basins and their protection. Consumption waste refers to products and materials that have lost their consumer properties due to physical or moral wear and tear. Industrial waste products include machines, equipment and other obsolete machinery from enterprises. Consumption waste consists of waste generated as a result of human activities and discarded by them as unwanted or useless. The morphological composition of consumption waste includes cardboard, various types of paper (newspapers, packaging or consumer papers), various types of containers (wooden, glass, metal), items and products made of wood, metal, leather, glass, plastic, textile and other materials that are either no longer in use or have lost their consumer properties. This encompasses broken or outdated household appliances – garbage, as well as agricultural and household food waste – called offal. A special category of waste, primarily industrial waste products, consists of radioactive waste generated during the extraction, production and use of radioactive substances as fuel for nuclear power plants, vehicles (i.e. nuclear submarines) and other purposes. Radioactive waste poses a significant environmental hazard. Toxic waste, including some initially non-hazardous waste that acquires toxic properties during storage, also poses a substantial threat to the environment (Borovsky, 2001).

In present paper, our focus is laid on an open landfill for solid consumption waste – a specially equipped area in the open air where municipal or private companies gather and transport organic and inorganic solid consumption waste, primarily consisting of household waste. Landfills for solid household waste are complexes of environmental structures designed for the storage, isolation and neutralization of solid household waste. They provide protection against the pollution of the atmosphere, soil, surface, and groundwater, as well as prevent the spread of rodents, insects and pathogenic microorganisms (Rao *et al*., 2017).

Nowadays every city generates a vast amount of waste, especially in the case of megalopolis. Some ages ago the composition of urban waste was predominantly organic, and their burial posed little problem and immediate threats. Organic waste undergoes natural decomposition in the environment and does not contain toxic substances. However, the development of civilization has altered the composition of consumption waste – more and more non-degradable components emerged, such as glass, ceramics, metals, rubber, plastics, as well as toxic substances like mercury, batteries, expired pharmaceuticals and so on. This has stipulated a shift in the technological approach to the disposal of solid consumption waste.

Urban landfills for solid consumption waste have specific rules in terms of design and organization (US EPA, 2023; RF MC, 1997): the landfill's service lifecycle ranges from 15 to 20 years; it should be located at least 1 km away from the nearest residential building; groundwater should be at a depth greater than two meters (the landfill should be equipped with wells for continuous monitoring of groundwater pollution levels); a protective forest belt of at least 20 meters wide should surround the landfill; the landfill cannot be situated in an area with water bodies (not only rivers and lakes but also springs); a good asphalt road should be laid to the landfill; special grids and screens must be installed on the landfill to prevent gaseous emissions into the atmosphere and so on. This also includes geological and hydrogeological, technological, sanitary and sanitary-protective, architectural and construction, electrical and techno-economic rules and requirements. In the process of meeting the above-mentioned requirements imposed on landfills for solid household waste, both during the site selection, design and construction phase, as well as during their operation, various scientific, technical, economic, engineering and other issues arise that need to be addressed. This paper explores some of these issues related to the operational phase of landfills for solid consumption waste. Specifically, scientific and technical problems arise in managing processes in landfills for solid household waste to enhance biogas extraction. These issues are associated with filtration processes in an isotropic layered porous medium, which is the nature of any operational landfill for solid consumption waste. It is worth noting that the authors of this paper were actively involved in addressing these issues in the years 2011-2013 during the implementation of the scientific and innovative project 2010/0301/2DP/2.1.1.1.0/10/APIA/VIAA/151: "*New technology and software development for biogas extraction processes optimization*". This project received financial support from the European Regional Development Fund (ERDF) and the Liepaja city council, Latvia. Within the frames of this project, a group of scientists from Liepaja University collaborated with the waste management company LLC "Liepājas RAS" to optimize the processes of biogas production. During the implementation of the project 2010/0301/2DP/2.1.1.1.0/10/APIA/VIAA/151, an automatic control system was invented (Guseynov *et al.*, 2013f), designed for continuous monitoring and maintenance of anaerobic fermentation processes occurring in the layers of the landfill for solid consumption waste. These processes depend on anaerobic bacteria, and under normal dump conditions, the development and life cycle of bacteria are not regulated. For example, during the hot days, as the temperature increases, the temperature in the waste layers also rises, leading to the disruption of bacterial colonies. As a result, biogas production stops, even though there are still unused waste materials capable of producing biogas (Guseynov & Aleksejeva, 2013a; 2013b; 2013e; Guseynov *et al.*, 2013a; 2013b; 2013c; 2013d; 2013e). This implies that the surrounding environment significantly influences biogas production processes even in the deep layers of the landfill for solid consumption waste. The developed automatic control system allows maintaining colonies of anaerobic bacteria, essential for biogas production, within the required temperature range. It utilizes a special substance (also invented within the framework of the implementation of the scientific and innovative project 2010/0301/2DP/2.1.1.1.0/10/APIA/VIAA/151; Table 1) and technical water in a controlled irrigation system (also created within the same project). In creating the automatic control system for anaerobic fermentation processes occurring in the layers of the landfill for solid consumption waste, the authors of this paper developed and investigated mathematical models (Tolpaev, 2004; Tolpaev & Paliev, 2007; Guseynov & Aleksejeva, 2013a; 2013b; 2013e; Guseynov *et al.*, 2013a; 2013b; 2013c; 2013d; 2013e; 2011; Rimshans *et al.*, 2011a; 2011b; 2011c), which were utilized to:

(a) Define acceptable ranges for changes in three crucial controlled parameters – pressure, moisture and temperature of moisture – in all layers of the landfill for solid consumption waste. When these parameter values approach threshold limits (lower and upper bounds of acceptable ranges), the automatic irrigation system is triggered until these parameter values normalize (Table 2 (a)-(c)).

(b) Determine optimal locations for placing continuous monitoring sensors for pressure, moisture, and temperature of moisture in all layers of the landfill for solid consumption waste (Figure 2); (Guseynov *et al.*, 2013f; Abdullaev, 1990).

(c) Identify optimal locations for installing biogas extraction collectors, ensuring both the maximization of biogas extraction and the minimization of biogas (carbon dioxide) leakage into the atmosphere.

(d) Establish the optimal spatial distribution of the morphological composition of waste in the landfill for solid consumption waste. Ensuring this distribution guarantees the long-lasting gas presence even of long-exploited landfill layers.

Here, it is worth noting that in Latvia, there are several relevant spot-on topics in biogas production. Firstly, there are concerns about low production efficiency and suboptimal production conditions. Secondly, there is a concern about the negative impact of the production process on the environment, particularly related to uncontrolled gas emissions into the atmosphere (Guseynov & Aleksejeva, 2013c; 2013d). Until now, in several countries worldwide, these issues have been theoretically and practically addressed through the creation of appropriate mathematical models, algorithms for their solution and conducting technological calculations.

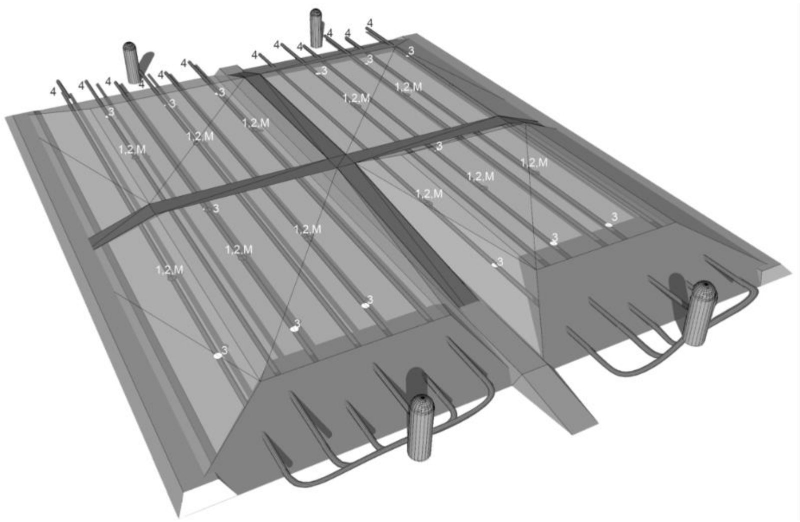

Figure 2. Schematic representation of the optimal placement of pressure, humidity and temperature sensors in the layers of the municipal solid waste landfill "Ķīvītes". Comprehensive information is available in Guseynov *et al.*, 2013f.

Table 1. Composition of substance for watering municipal solid waste landfill: * – can be combined in arbitrary proportions with substances marked *; ** – can be combined in arbitrary proportions with substances marked **; *** – can be combined in arbitrary proportions with substances marked ***.

Substance components	Fraction
Potassium chloride	1(kg)/10
Calcium sulphate	1(kg)/5
Potassium-magnesium sulphate	1(kg)/5
Calcium lactate	1(kg)/10
Ethyl acetate* *or* sodium acetate* *or* Ammonium acetate* *or* Lead(II) acetate*	1(kg)/10
Propionic acid	1(kg)/10
Butyric acid *(salts or esters of butyric acid)*	1(kg)/10
Methyl formate** *or* Sodium formate** *or* Ethyl formate** *or* Ammonium formate** *or* Triethyl orthoformate**	1(kg)/10
Process water	n (l):1(kg), n$\in[40, 70]$
Acetic acid*** *or* Oxalic acid*** *or* Citric acid***	100 (g) in 10 (l) substance

Table 2 (a). The watering system is triggered if the humidity, temperature and pressure sensors show the following information, respectively: for humidity, %

	Vertical distance from the surface of the landfill "Ķīvītes" (depth, m)	**Horizontal distance from the location of gas collection wells (length, m)**	**Months: 4, 9, 10**	**Months: 5, 6, 7, 8**
Humidity, %	$d_{\text{vertical}} \in [0,1]$	$l_{\text{horizontal}} \in [0,12]$	$H \le 65\% \cdot H_{\text{Normal}}$	$H \le 80\% \cdot H_{\text{Normal}}$
	$d_{\text{vertical}} \in [0,1]$	$l_{\text{horizontal}} \in (12,25]$	$H \le 65\% \cdot H_{\text{Normal}}$	$H \le 80\% \cdot H_{\text{Normal}}$
	$d_{\text{vertical}} \in [0,1]$	$l_{\text{horizontal}} \in (25,100]$	$H \le 65\% \cdot H_{\text{Normal}}$	$H \le 80\% \cdot H_{\text{Normal}}$
	$d_{\text{vertical}} \in (1,3]$	$l_{\text{horizontal}} \in [0,12]$	$H \le 80\% \cdot H_{\text{Normal}}$	$H \le 85\% \cdot H_{\text{Normal}}$
	$d_{\text{vertical}} \in (1,3]$	$l_{\text{horizontal}} \in (12,25]$	$H \le 80\% \cdot H_{\text{Normal}}$	$H \le 85\% \cdot H_{\text{Normal}}$
	$d_{\text{vertical}} \in (1,3]$	$l_{\text{horizontal}} \in (25,100]$	$H \le 80\% \cdot H_{\text{Normal}}$	$H \le 85\% \cdot H_{\text{Normal}}$

	$d_{\text{vertical}} \in (3,6]$	$l_{\text{horizontal}} \in [0,12]$	$H \le 85\% \cdot H_{\text{Normal}}$	$H \le 90\% \cdot H_{\text{Normal}}$
	$d_{\text{vertical}} \in (3,6]$	$l_{\text{horizontal}} \in (12,25]$	$H \le 85\% \cdot H_{\text{Normal}}$	$H \le 90\% \cdot H_{\text{Normal}}$
	$d_{\text{vertical}} \in (3,6]$	$l_{\text{horizontal}} \in (25,100]$	$H \le 85\% \cdot H_{\text{Normal}}$	$H \le 90\% \cdot H_{\text{Normal}}$
	$d_{\text{vertical}} \in (6,9]$	$l_{\text{horizontal}} \in [0,12]$	$H \le 85\% \cdot H_{\text{Normal}}$	$H \le 90\% \cdot H_{\text{Normal}}$
	$d_{\text{vertical}} \in (6,9]$	$l_{\text{horizontal}} \in (12,25]$	$H \le 85\% \cdot H_{\text{Normal}}$	$H \le 90\% \cdot H_{\text{Normal}}$
	$d_{\text{vertical}} \in (6,9]$	$l_{\text{horizontal}} \in (25,100]$	$H \le 85\% \cdot H_{\text{Normal}}$	$H \le 90\% \cdot H_{\text{Normal}}$

Table 2 (b). The watering system is triggered if the humidity, temperature and pressure sensors show the following information, respectively: for temperature, °C

	Vertical distance from the surface of the landfill "Ķīvītes" (depth, m)	**Horizontal distance from the location of gas collection wells (length, m)**	**Months: 1, 2, 11, 12**	**Months: 3, 4, 9, 10**	**Months: 5, 6, 7, 8**
Temperature, °C	$d_{\text{vertical}} \in [0,1]$	$l_{\text{horizontal}} \in [0,12]$	$T \in [17,25]$	$T \in [19,30]$	$T \in [22,34]$
	$d_{\text{vertical}} \in [0,1]$	$l_{\text{horizontal}} \in (12,25]$	$T \in [17,23]$	$T \in [19,31]$	$T \in [20,33]$
	$d_{\text{vertical}} \in [0,1]$	$l_{\text{horizontal}} \in (25,100]$	$T \in [17,23]$	$T \in [17,33]$	$T \in [19,34]$
	$d_{\text{vertical}} \in (1,3]$	$l_{\text{horizontal}} \in [0,12]$	$T \in [28,34]$	$T \in [29,36]$	$T \in [31,37]$
	$d_{\text{vertical}} \in (1,3]$	$l_{\text{horizontal}} \in (12,25]$	$T \in [26,35]$	$T \in [27,37]$	$T \in [30,40]$
	$d_{\text{vertical}} \in (1,3]$	$l_{\text{horizontal}} \in (25,100]$	$T \in [26,35]$	$T \in [27,37]$	$T \in [30,40]$
	$d_{\text{vertical}} \in (3,6]$	$l_{\text{horizontal}} \in [0,12]$	$T \in [31,42]$	$T \in [32,43]$	$T \in [33,44]$
	$d_{\text{vertical}} \in (3,6]$	$l_{\text{horizontal}} \in (12,25]$	$T \in [30,43]$	$T \in [31,44]$	$T \in [32,45]$
	$d_{\text{vertical}} \in (3,6]$	$l_{\text{horizontal}} \in (25,100]$	$T \in [30,42]$	$T \in [30,43]$	$T \in [31,47]$
	$d_{\text{vertical}} \in (6,9]$	$l_{\text{horizontal}} \in [0,12]$	$T \in [41,46]$	$T \in [42,48]$	$T \in [42,53]$
	$d_{\text{vertical}} \in (6,9]$	$l_{\text{horizontal}} \in (12,25]$	$T \in [41,45]$	$T \in [41,49]$	$T \in [41,52]$
	$d_{\text{vertical}} \in (6,9]$	$l_{\text{horizontal}} \in (25,100]$	$T \in [41,45]$	$T \in [41,49]$	$T \in [41,52]$

Table 2 (c). The watering system is triggered if the humidity, temperature and pressure sensors show the following information, respectively: for pressure, Pa.

	Vertical distance from the surface of the landfill "Ķīvītes" (depth, m)	**Horizontal distance from the location of gas collection wells (length, m)**	**Months: 1÷12**
Pressure, Pa	$d_{\text{vertical}} \in [0,1]$	$l_{\text{horizontal}} \in [0,12]$	$\Delta P \approx 2360$
	$d_{\text{vertical}} \in [0,1]$	$l_{\text{horizontal}} \in (12,25]$	$\Delta P \in [2200,3350]$
	$d_{\text{vertical}} \in [0,1]$	$l_{\text{horizontal}} \in (25,100]$	$\Delta P \in [2200,4000]$
	$d_{\text{vertical}} \in (1,3]$	$l_{\text{horizontal}} \in [0,12]$	$\Delta P \approx 6040$
	$d_{\text{vertical}} \in (1,3]$	$l_{\text{horizontal}} \in (12,25]$	$\Delta P \in [5870,8000]$
	$d_{\text{vertical}} \in (1,3]$	$l_{\text{horizontal}} \in (25,100]$	$\Delta P \in [6000,9000]$
	$d_{\text{vertical}} \in (3,6]$	$l_{\text{horizontal}} \in [0,12]$	$\Delta P \approx 12110$
	$d_{\text{vertical}} \in (3,6]$	$l_{\text{horizontal}} \in (12,25]$	$\Delta P \in [11780,14350]$

	$d_{\text{vertical}} \in (3,6]$	$l_{\text{horizontal}} \in (25,100]$	$\Delta P \in [12000,15000]$
	$d_{\text{vertical}} \in (6,9]$	$l_{\text{horizontal}} \in [0,12]$	$\Delta P \approx 12110$
	$d_{\text{vertical}} \in (6,9]$	$l_{\text{horizontal}} \in (12,25]$	$\Delta P \in [11780,14350]$
	$d_{\text{vertical}} \in (6,9]$	$l_{\text{horizontal}} \in (25,100]$	$\Delta P \in [12000,15000]$

There is a design formula $\Delta P = g \cdot h \cdot (\rho_{\text{outer air}} - \rho_{\text{biogas}})$, where ΔP $(\text{unit of measure: Pa})$ is the design pressure (i.e. the available pressure); $g \approx 9.80665$ $(\text{unit of measure}: m/c^2)$ is the acceleration of free fall; h $(\text{unit of measure}: m)$ is the height that determines the calculated pressure drop in the gas collection well, equal to the sum of the average thickness of the waste layer and the distance to the mouth of the gas outlet above the surface of the landfill; $\rho_{\text{outer air}}$ $(\text{unit of measure}: kg/m^3)$ is the calculated density of outside air; ρ_{biogas} $(\text{unit of measure}: kg/m^3)$ is the calculated density of biogas.

The main value of the scientific and innovative project 2010/0301/2DP/2.1.1.1.0/10/APIA/VIAA/151 lay in connecting theoretical concepts with practical experiments in the conditions of the open landfill "Ķīvītes" in the Grobiņa district of the Kurzeme region in Latvia, which serves as the technological base of LLC "Liepājas RAS". The developed new technologies and software for optimizing biogas extraction processes, while adhering to the fundamental physico-chemical processes of biogas extraction, contributed to increasing production efficiency and reducing the negative impact of the production process on the environment. The most significant and complex scientific goal of the scientific and innovative project 2010/0301/2DP/2.1.1.1.0/10/APIA/VIAA/151 was successfully achieved: the simultaneous integration and interconnection of multiple factors – mathematical and physical methods, physico-chemical processes, information technologies, and practical activities through field experiments. To achieve this complex goal, first and foremost, a mathematical computational model of physico-chemical processes was developed. This model enables the adherence of the morphological composition of waste and its spatial distribution, along with the necessary conditions of temperature, pressure and moisture. Secondly, algorithms for solving this mathematical computational model and software were developed using parallel data processing technology. The aim was to collect and systematize the data obtained from measurements at the landfill for solid consumption waste. Thirdly, throughout the entire implementation period of the scientific and innovative project 2010/0301/2DP/2.1.1.1.0/10/APIA/VIAA/151, field studies were conducted. The installation of necessary equipment for collecting measurement data on the landfill site was carried out, along with the systematization and analysis of the gathered data. The conducted research in Latvia was distinctive from a scientific perspective, as it involved describing complex physical processes in a specific environment using theoretical methods. Scientifically justified recommendations for optimizing all these processes were developed. Prior to this project, such an approach had not been practiced in Latvia. To conclude this section, let's briefly describe the scientific group of this project. The scientific group of this project consisted of members from the Institute of Mathematical Sciences and Information Technologies, including the project's scientific leader, four senior researchers, six researchers, a mechanical engineer and five technical staff members. The team has successfully enhanced their knowledge and professionalism through traineeships at foreign universities and by participating in global scientific conferences on the development of mathematical physics, analytical and numerical methods and high-performance solutions in parallel data processing technologies. This experience proved to be highly beneficial in achieving the planned goals of the project.

2. Mathematical model (direct problem) of the linear fluid filtration process in an isotropic porous medium

Processes of fluid filtration occur in porous media, which, depending on their physico-chemical-mechanical properties, can be classified as isotropic materials (meaning that filtration properties at each point in the medium are the same in all directions, i.e. the corresponding functions in mathematical models are scalar functions) or anisotropic materials (meaning that filtration properties at each point in the medium vary in different directions and consequently, the corresponding functions in mathematical models are vector functions). Moreover, layers and/or stratums of porous media are divided into two parts – "active/productive" and "inactive/non-productive" layers. Moreover, the "active/productive" layers and/or stratums in real processes (objects, phenomena) are always curved and have variable thickness and not only demonstrating isotropic or anisotropic, homogeneous or heterogeneous filtration properties.

Let us assume that the non-uniform porous region under investigation, which is a truncated pyramid, possesses an isotropic and spatially periodic (not necessarily with a constant period) structure. The primary structural element of this structure is considered to be a rectangular parallelepiped. Let's represent the permeability coefficient in a multiplicative form

$$k(x,y,z)=K(\alpha,\beta,\gamma)=K^{\{1\}}\left(\alpha(x,y,z)\right)\cdot K^{\{2\}}\left(\beta(x,y,z)\right)\cdot K^{\{3\}}\left(\gamma(x,y,z)\right), \quad (1)$$

Where the auxiliary argument-functions $\alpha=\alpha(x,y,z)$, $\beta=\beta(x,y,z)$, $\gamma=\gamma(x,y,z)$, firstly define the geometry of the periodic structure of the porous medium. The periods α, β and γ represent the dimensions of the repeating structural elements, which are rectangular parallelepipeds assumed to constitute the porous truncated pyramid mentioned earlier. Secondly, these functions satisfy the following conditions:

(a) $$\langle\nabla\alpha,\nabla\beta\rangle=\langle\nabla\alpha,\nabla\gamma\rangle=\langle\nabla\beta,\nabla\gamma\rangle\equiv 0, \quad (2)$$

where $\langle\bullet,\bullet\rangle$ denotes the scalar derivative;

(b) $$\max\left\{\int\limits_{\alpha(x,y,z)}^{\alpha(x,y,z)+T_{per.}(\alpha(x,y,z))}\left|\frac{\partial}{\partial\alpha(x,y,z)}\left(x(\alpha,\beta,\gamma)\cdot\vec{i}_1+\right.\right.\right.$$
$$\left.\left.\left.+y(\alpha,\beta,\gamma)\cdot\vec{i}_2+z(\alpha,\beta,\gamma)\cdot\vec{i}_3\right)\right|\right\}d\alpha,$$
$$\int\limits_{\beta(x,y,z)}^{\beta(x,y,z)+T_{per.}(\beta(x,y,z))}\left|\frac{\partial}{\partial\beta(x,y,z)}\left[x(\alpha,\beta,\gamma)\cdot\vec{i}_1+y(\alpha,\beta,\gamma)\cdot\vec{i}_2+z(\alpha,\beta,\gamma)\cdot\vec{i}_3\right]\right|d\beta,$$

$$\int_{\gamma(x,y,z)}^{\gamma(x,y,z)+T_{per.}(\gamma(x,y,z))} \left| \frac{\partial}{\partial \gamma(x,y,z)} \Big(x(\alpha,\beta,\gamma) \cdot \vec{i}_1 + y(\alpha,\beta,\gamma) \cdot \vec{i}_2 + \right.$$
$$\left. + z(\alpha,\beta,\gamma) \cdot \vec{i}_3 \Big) \right| d\gamma \Big\} \ll L. \qquad (3)$$

Here $T_{per.}(\omega(x,y,z))$ serves as the period of the argument-function ω, $\omega = \{\alpha;\beta;\gamma\}$, besides, if along a specific direction $\omega = \{\alpha;\beta;\gamma\}$ the permeability $k(x,y,z)$ does not change, then period $T_{per.}(\omega(x,y,z))$ can be chosen as equal to $T_{per.}(\bar{\omega})$, $\bar{\omega} = \{\alpha;\beta;\gamma\}/\omega$; L is the characteristic size of the filtration region; $\omega(\alpha,\beta,\gamma) = \sigma^{-1}(x,y,z)$, $\omega = \{x;y;z\}$, $\sigma = (\alpha;\beta;\gamma)$.

(c)
$$\min \left\{ \int_{\alpha(x,y,z)}^{\alpha(x,y,z)+T_{per.}(\alpha(x,y,z))} \left| \frac{\partial}{\partial \alpha(x,y,z)} \Big(x(\alpha,\beta,\gamma) \cdot \vec{i}_1 + \right. \right.$$
$$\left. \left. + y(\alpha,\beta,\gamma) \cdot \vec{i}_2 + z(\alpha,\beta,\gamma) \cdot \vec{i}_3 \Big) \right| \right\} d\alpha,$$
$$\int_{\beta(x,y,z)}^{\beta(x,y,z)+T_{per.}(\beta(x,y,z))} \left| \frac{\partial}{\partial \beta(x,y,z)} \Big[x(\alpha,\beta,\gamma) \cdot \vec{i}_1 + y(\alpha,\beta,\gamma) \cdot \vec{i}_2 + z(\alpha,\beta,\gamma) \cdot \vec{i}_3 \Big] \right| d\beta,$$
$$\int_{\gamma(x,y,z)}^{\gamma(x,y,z)+T_{per.}(\gamma(x,y,z))} \left| \frac{\partial}{\partial \gamma(x,y,z)} \Big(x(\alpha,\beta,\gamma) \cdot \vec{i}_1 + y(\alpha,\beta,\gamma) \cdot \vec{i}_2 + \right.$$
$$\left. + z(\alpha,\beta,\gamma) \cdot \vec{i}_3 \Big) \right| d\gamma \Big\} \gg \max\{\Delta_1, \Delta_2, \Delta_3\}. \qquad (4)$$

Here Δ_i $(i = \overline{1,3})$ is the distance between the ends of the arc along the $\omega = \{\alpha;\beta;\gamma\}$-th coordinate line and the end of corresponding i-th $(i = \overline{1,3})$ edge of the structural element, represented by a rectangular parallelepiped, i.e. if to consider point $O(x,y,z)$ of each structural element, rectangular parallelepiped to be a local reference point and denote with points $A_i(x,y,z)$ $(i = \overline{1,3})$ the ends of its three edges OA_i $(i = \overline{1,3})$ for this rectangular parallelepiped, and with points $\bar{A}_i(\alpha,\beta,\gamma)$ $(i = \overline{1,3})$ – the ends of the three arcs $O\bar{A}_i$ $(i = \overline{1,3})$ of the corresponding curvilinear parallelepiped, then the value Δ_i $(i = \overline{1,3})$ is determined as $\Delta_i \overset{def}{\equiv} |A_i - \bar{A}_i|$ $(i = \overline{1,3})$.

Condition (a) means that the level surfaces $\alpha(x,y,z) = \alpha_0 \equiv const.$, $\beta(x,y,z) = \beta_0 \equiv const.$ and $\gamma(x,y,z) = \gamma_0 \equiv const.$ in the porous truncated pyramid form a mutually orthogonal system of surfaces; condition (b) means that within the filtration region, the length

$$\int_{\omega(x,y,z)}^{\omega(x,y,z)+T_{per.}(\omega(x,y,z))} \left| \frac{\partial}{\partial \omega(x,y,z)} \Big[x(\alpha,\beta,\gamma) \cdot \vec{i}_1 + y(\alpha,\beta,\gamma) \cdot \vec{i}_2 + z(\alpha,\beta,\gamma) \cdot \vec{i}_3 \Big] \right| d\omega$$

of the arc for $\omega = \{\alpha;\beta;\gamma\}$-th coordinate line, corresponding to the period $T_{per.}(\omega)$, $\omega=\{\alpha;\beta;\gamma\}$, is infinitely small compared to its characteristic size L; condition (c) means that in the curvilinear parallelepiped, which is bounded by coordinate surfaces $\alpha=\alpha_0\equiv const.$ and $\alpha=\alpha_0+T_{per.}(\alpha(x,y,z))$; $\beta=\beta_0\equiv const.$ and $\beta=\alpha_0+T_{per.}(\alpha(x,y,z))$; $\gamma=\alpha_0\equiv const.$ and $\gamma=\gamma_0+T_{per.}(\gamma(x,y,z))$, the ends of arcs with a length

$$\int\limits_{\omega(x,y,z)}^{\omega(x,y,z)+T_{per.}(\omega(x,y,z))}\left|\frac{\partial}{\partial\omega(x,y,z)}\left[x(\alpha,\beta,\gamma)\cdot\vec{i}_1+y(\alpha,\beta,\gamma)\cdot\vec{i}_2+z(\alpha,\beta,\gamma)\cdot\vec{i}_3\right]\right|d\omega$$

Deviate insignificantly from the ends of the corresponding tangent lines (or more precisely, segments).

The simplifying conditions (2)-(4) allow us to consider an elementary curvilinear parallelepiped as an elementary rectangular parallelepiped with edges OA_i $\left(i=\overline{1,3}\right)$, the lengths of which are equal to:

$$L(\omega)\overset{def}{\equiv}\int\limits_{\omega(x,y,z)}^{\omega(x,y,z)+T_{per.}(\omega(x,y,z))}\left|\frac{\partial}{\partial\omega(x,y,z)}\left(x(\alpha,\beta,\gamma)\cdot\vec{i}_1+y(\alpha,\beta,\gamma)\cdot\vec{i}_2+\right.\right.$$
$$\left.\left.+z(\alpha,\beta,\gamma)\cdot\vec{i}_3\right)\right|d\omega,\ \omega=\{\alpha;\beta;\gamma\}. \tag{5}$$

We will call such an elementary rectangular parallelepiped an elementary approximation-averaging parallelepiped.

It is necessary to construct such a simplified mathematical model that describes the basic filtration properties of isotropic media with a permeability coefficient determined by formula (1). In this model, the functional coefficients in the filtration equations are continuous functions of spatial variables and they are not necessarily periodic. The general model of fluid filtration in inhomogeneous isotropic media with periodically varying permeability includes filtration equations, the functional coefficients of which are rapidly oscillating functions and in general, can be discontinuous piecewise-continuous functions. Therefore, finding an analytical solution to the corresponding initial-boundary value problems for such equations is challenging (Kochina, 1977, Kochina *et al.*, 1994; Kochina & Kochina, 1996).

Since, with respect to all three one-dimensional flows that are perpendicular to the level surfaces $\alpha(x,y,z)=\alpha_0\equiv const.$, $\beta(x,y,z)=\beta_0\equiv const.$ and $\gamma(x,y,z)=\gamma_0\equiv const.$, the porous truncated pyramid with permeability (1) has different filtration characteristics, when modelling, the approximating porous curvilinear layered truncated pyramid must be replaced with a "fictitious" anisotropic medium that should completely preserve identical filtration characteristics with respect to all three mentioned flows (in general, not one-dimensional any longer). Depending on the level of approximation of these filtration flows – at the level of a structural element (i.e. within the boundaries of elementary approximating-averaging parallelepiped) or at the level of the entire filtration area – we can talk about approximation methods, namely, Ollendorff's method of local homogeneous-anisotropic approximation (Ollendorff, 1932; Kochina, 1977; Emikh, 1984; Tolpaev, 2004; Ollendorff, 1932) or Leibenzon's method of integral homogeneous-anisotropic approximation (Leibenzon, 1947; Kochina, 1977; Collins, 1961; Barenblatt *et al.*, 1984; Bear *et. al.*, 1968; Emikh, 1984; Tolpaev, 2004; Golubev & Tumashev, 1972). Note that the methods of local and integral homogeneous-anisotropic approximations lead to different permeability values in anisotropic models along the $\omega = \{\alpha; \beta; \gamma\}$ coordinate lines (Kochina, 1977; Bear *et. al.*, 1968).

Given the assumptions above, let's apply the Ollendorff method to our "pyramidal" problem to build a filtration model in a porous curved-layered medium with spatially varying permeability, the coefficient of which is defined by formula (1). To do this, let's compare all three one-dimensional flows that are perpendicular to the level surfaces $\alpha(x,y,z) = \alpha_0 \equiv const.$, $\beta(x,y,z) = \beta_0 \equiv const.$, $\gamma(x,y,z) = \gamma_0 \equiv const.$, in the elementary approximating-averaging parallelepiped (elementary structural element), filled with fluids (heterogeneous medium) with permeability coefficients of the form (1), and in the same elementary approximating-averaging parallelepiped, filled with a homogeneous medium with rectilinear anisotropy. Due to condition/assumption (3), it can be assumed that within the elementary structural unit (once again, let's recall that the elementary structural unit is an elementary approximating-averaging parallelepiped) density $\rho(\alpha,\beta,\gamma;t)$, dynamic viscosity $\mu(\alpha,\beta,\gamma;t)$, temperature $T(\alpha,\beta,\gamma;t)$ and other physical characteristics of the fluid are constant, and moreover, the filtration process within this elementary structural unit occurs instantaneously, i.e. all physical characteristics of the fluid within the elementary approximating-averaging parallelepiped do not depend on time (a more precise mathematical interpretation of this statement/assumption using the Dirac delta function is given in Pilatovsky, 1966; Tolpaev & Paliev, 2007; Aravin, 1974). Hence, in the local Cartesian coordinate system $\alpha \times \beta \times \gamma$ (locality is understood in the sense that for each elementary structural unit, its own Cartesian coordinate system is introduced) we can write the following well-known fluid filtration equations (Kochina, 1977; Golubev & Tumashev, 1972; Aravin & Numerov, 1953; Scheidegger, 1974; Basniev *et al.*, 2006):

$$\frac{\partial}{\partial\alpha}\left\{K(\alpha,\beta,\gamma)\cdot\frac{\partial p(\alpha,\beta,\gamma)}{\partial\alpha}\right\}+\frac{\partial}{\partial\beta}\left\{K(\alpha,\beta,\gamma)\cdot\frac{\partial p(\alpha,\beta,\gamma)}{\partial\beta}\right\}+ \\ +\frac{\partial}{\partial\gamma}\left\{K(\alpha,\beta,\gamma)\cdot\frac{\partial p(\alpha,\beta,\gamma)}{\partial\gamma}\right\}=0, \tag{6}$$

where $p(\alpha,\beta,\gamma)$ is the pressure;

$$\vec{\vartheta} = -\frac{K(\alpha,\beta,\gamma)}{\mu(\alpha,\beta,\gamma)}\cdot\nabla p, \ \textit{(Darcy's Law)} \tag{7}$$

Where $\vec{\vartheta} = \vec{\vartheta}(\alpha,\beta,\gamma)$ defines the filtration velocity field within the elementary approximated-averaged parallelepiped;

Then, for the one-dimensional filtration flow that occurs along the $\alpha-$coordinate line within the elementary approximated-averaged parallelepiped, we can write from (6):

$$\frac{\partial}{\partial\alpha}\left\{K^{\{1\}}(\alpha)\cdot\frac{\partial p(\alpha,\beta,\gamma)}{\partial\alpha}\right\}=0,$$

Which, after integration, yields:

$$p(\alpha,\beta,\gamma)=\left[K^{\{1\}}(\alpha)\cdot\frac{\partial p(\alpha,\beta,\gamma)}{\partial\gamma}\right]\Bigg|_{\alpha=0}\cdot\int_0^{\alpha}\frac{d\alpha_1}{K^{\{1\}}(\alpha_1)}+p(\alpha,\beta,\gamma)\big|_{\alpha=0}.$$

Assuming that the function $p(\alpha,\beta,\gamma)\big|_{\alpha=L(\alpha)}$ is given, from the last formula we can determine that:

$$\left[K^{\{1\}}(\alpha)\cdot\frac{\partial p(\alpha,\beta,\gamma)}{\partial\gamma}\right]\Bigg|_{\alpha=0}=\frac{p(\alpha,\beta,\gamma)\big|_{\alpha=L(\alpha)}+p(\alpha,\beta,\gamma)\big|_{\alpha=0}}{\int\limits_0^{L(\alpha)}\frac{d\alpha_1}{K^{\{1\}}(\alpha_1)}}.$$

So, we have obtained that:

$$p(\alpha,\beta,\gamma)=\frac{p(\alpha,\beta,\gamma)\big|_{\alpha=L(\alpha)}+p(\alpha,\beta,\gamma)\big|_{\alpha=0}\cdot\left[\int\limits_0^{\alpha}\frac{d\alpha_1}{K^{\{1\}}(\alpha_1)}-\int\limits_0^{L(\alpha)}\frac{d\alpha_1}{K^{\{1\}}(\alpha_1)}\right]}{\int\limits_0^{L(\alpha)}\frac{d\alpha_1}{K^{\{1\}}(\alpha_1)}},\qquad(8)$$

Where $L(\alpha)=|OA_1|$ is determined by formula (5) and represents the length of the edge OA_1 of the elementary approximating-averaging parallelepiped (recall that, according to the above construction/assumption, the edge OA_1 lies on the α-coordinate line; edge OA_2 – on the β-coordinate line; edge OA_3 – on the γ-coordinate line); $K^{\{1\}}(\bullet)$ is a function from (1).

The total fluid filtration flow passing through the lateral surface of the elementary approximating-averaging parallelepiped, perpendicular to the α-coordinate line, can be calculated using the formula

$$Q_\alpha\overset{def}{\equiv}-\frac{p(\alpha,\beta,\gamma)\big|_{\alpha=L(\alpha)}-p(\alpha,\beta,\gamma)\big|_{\alpha=0}}{\int\limits_0^{L(\alpha)}\frac{d\alpha_1}{K^{\{1\}}(\alpha_1)}}\cdot\int\limits_0^{L(\beta)}K^{\{2\}}(\beta_1)\,d\beta_1\int\limits_0^{L(\gamma)}\frac{K^{\{3\}}(\gamma_1)}{\mu(\alpha,\beta_1,\gamma_1)}d\gamma_1.$$

By repeating this reasoning, we can determine the total fluid filtration flows passing through the other two lateral surfaces of the elementary approximating-averaging parallelepiped, which are perpendicular to the β-coordinate line and the γ-coordinate line, respectively:

$$Q_\beta\overset{def}{\equiv}-\frac{p(\alpha,\beta,\gamma)\big|_{\beta=L(\beta)}-p(\alpha,\beta,\gamma)\big|_{\beta=0}}{\int\limits_0^{L(\beta)}\frac{d\beta_1}{K^{\{2\}}(\beta_1)}}\cdot\int\limits_0^{L(\alpha)}K^{\{1\}}(\alpha_1)\,d\alpha_1\int\limits_0^{L(\gamma)}\frac{K^{\{3\}}(\gamma_1)}{\mu(\alpha_1,\beta,\gamma_1)}d\gamma_1,$$

$$Q_\gamma\overset{def}{\equiv}-\frac{p(\alpha,\beta,\gamma)\big|_{\gamma=L(\gamma)}-p(\alpha,\beta,\gamma)\big|_{\gamma=0}}{\int\limits_0^{L(\gamma)}\frac{d\gamma_1}{K^{\{3\}}(\gamma_1)}}\cdot\int\limits_0^{L(\alpha)}K^{\{1\}}(\alpha_1)\,d\alpha_1\int\limits_0^{L(\beta)}\frac{K^{\{2\}}(\beta_1)}{\mu(\alpha_1,\beta_1,\gamma)}d\beta_1.$$

Indeed, due to our assumption that within the elementary approximating-averaging parallelepiped, the dynamic viscosity of the fluid is constant (along with other physical characteristics of the fluid), in the last three formulas $\mu(\alpha,\beta,\gamma)=\mu_{const.}\equiv const.$ and, therefore, this value is taken out of the integral expressions. As a result, the multiple integrals in the right-hand sides of these formulas transform into repeated integrals:

$$Q_\alpha \overset{def}{\equiv} -\frac{p(\alpha,\beta,\gamma)\big|_{\alpha=L(\alpha)} - p(\alpha,\beta,\gamma)\big|_{\alpha=0}}{\mu_{const.}\cdot\int\limits_0^{L(\alpha)}\frac{d\alpha_1}{K^{\{1\}}(\alpha_1)}}\cdot\int\limits_0^{L(\beta)}K^{\{2\}}(\beta_1)\,d\beta_1\cdot\int\limits_0^{L(\gamma)}K^{\{3\}}(\gamma_1)\,d\gamma_1, \quad (9)$$

$$Q_\beta \overset{def}{\equiv} -\frac{p(\alpha,\beta,\gamma)\big|_{\beta=L(\beta)} - p(\alpha,\beta,\gamma)\big|_{\beta=0}}{\mu_{const.}\cdot\int\limits_0^{L(\beta)}\frac{d\beta_1}{K^{\{2\}}(\beta_1)}}\cdot\int\limits_0^{L(\alpha)}K^{\{1\}}(\alpha_1)\,d\alpha_1\cdot\int\limits_0^{L(\gamma)}K^{\{3\}}(\gamma_1)\,d\gamma_1, \quad (10)$$

$$Q_\gamma \overset{def}{\equiv} -\frac{p(\alpha,\beta,\gamma)\big|_{\gamma=L(\gamma)} - p(\alpha,\beta,\gamma)\big|_{\gamma=0}}{\mu_{const.}\cdot\int\limits_0^{L(\gamma)}\frac{d\gamma_1}{K^{\{3\}}(\gamma_1)}}\cdot\int\limits_0^{L(\alpha)}K^{\{1\}}(\alpha_1)\,d\alpha_1\cdot\int\limits_0^{L(\beta)}K^{\{2\}}(\beta_1)\,d\beta_1. \quad (11)$$

As shown in Kochina (1977); Aravin & Numerov (1953); Scheidegger (1974), in the local Cartesian coordinate system $\alpha\times\beta\times\gamma$ within a small volume (in our case, this is the elementary approximating-averaging parallelepiped), the permeabilities of the linearly anisotropic porous medium along local Cartesian axes can be considered constant: $k_\alpha \equiv const.,\ k_\beta \equiv const.,\ k_\gamma \equiv const.$. Then, the fluid filtration equation within the small volume of this linearly anisotropic porous medium has the following form (Kochina, 1977; Aravin & Numerov, 1953; Scheidegger, 1974):

$$\sum_{i=1}^{3} k_{\omega_i}\cdot\frac{\partial^2 p(\omega_1,\omega_2,\omega_3)}{\partial\omega_i^2} = 0;\ \{\omega_1;\omega_2;\omega_3\} \overset{def}{\equiv} \{\alpha;\beta;\gamma\}. \quad (12)$$

In this case, the fluid filtration velocity field is determined by Darcy's tensor law:

$$\vartheta_{\omega_i} = -\frac{k_{\omega_i}}{\mu_{const.}}\cdot\frac{\partial p}{\partial\omega_i};\ \{\omega_1;\omega_2;\omega_3\} \overset{def}{\equiv} \{\alpha;\beta;\gamma\}. \quad (13)$$

For a one-dimensional filtration flow that occurs along the α- coordinate line within the elementary approximating-averaging parallelepiped, from (12) we obtain that

$$p(\alpha,\beta,\gamma) = \frac{p(\alpha,\beta,\gamma)\big|_{\alpha=L(\alpha)} - p(\alpha,\beta,\gamma)\big|_{\alpha=0}}{L(\alpha)}\cdot\alpha + p(\alpha,\beta,\gamma)\big|_{\alpha=0}.$$

Therefore, we can calculate the total filtration flow of the fluid that passes through the lateral surface of the elementary averaged parallelepiped, perpendicular to the $\alpha-$ coordinate line.

$$Q_\alpha \overset{def}{\equiv} -\frac{k_\alpha}{\mu_{const.}}\cdot\int\limits_0^{L(\beta)}d\beta_1\int\limits_0^{L(\gamma)}\frac{\partial p(\alpha,\beta_1,\gamma_1)}{\partial\alpha}d\gamma_1 =$$

$$= -\frac{k_\alpha}{\mu_{const.}}\cdot\int\limits_0^{L(\beta)}d\beta_1\int\limits_0^{L(\gamma)}\left\{\frac{p(\alpha,\beta,\gamma)\big|_{\alpha=L(\alpha)} - p(\alpha,\beta,\gamma)\big|_{\alpha=0}}{L(\alpha)}\right\}d\gamma_1 =$$

$$= -\frac{k_\alpha}{\mu_{const.}}\cdot\frac{L(\beta)\cdot L(\gamma)}{L(\alpha)}\cdot\left\{p(\alpha,\beta,\gamma)\big|_{\alpha=L(\alpha)} - p(\alpha,\beta,\gamma)\big|_{\alpha=0}\right\}. \quad (14)$$

Absolutely similarly, we can determine the total filtration flows of the fluid passing through the other two lateral surfaces of the elementary averaged parallelepiped, which are perpendicular to the respective $\beta-$ coordinate lines and to the $\gamma-$ coordinate lines:

$$Q_\beta \overset{def}{\equiv} -\frac{k_\beta}{\mu_{const.}} \cdot \int\limits_0^{L(\alpha)} d\alpha_1 \int\limits_0^{L(\gamma)} \frac{\partial p(\alpha_1,\beta,\gamma_1)}{\partial\beta} d\gamma_1 =$$

$$= -\frac{k_\beta}{\mu_{const.}} \cdot \int\limits_0^{L(\alpha)} d\alpha_1 \int\limits_0^{L(\gamma)} \left\{ \frac{p(\alpha,\beta,\gamma)\big|_{\beta=L(\beta)} - p(\alpha,\beta,\gamma)\big|_{\beta=0}}{L(\beta)} \right\} d\gamma_1 =$$

$$= -\frac{k_\beta}{\mu_{const.}} \cdot \frac{L(\alpha)\cdot L(\gamma)}{L(\beta)} \cdot \left\{ p(\alpha,\beta,\gamma)\big|_{\beta=L(\beta)} - p(\alpha,\beta,\gamma)\big|_{\beta=0} \right\}, \tag{15}$$

$$Q_\gamma \overset{def}{\equiv} -\frac{k_\gamma}{\mu_{const.}} \cdot \int\limits_0^{L(\alpha)} d\alpha_1 \int\limits_0^{L(\beta)} \frac{\partial p(\alpha_1,\beta_1,\gamma)}{\partial\gamma} d\beta_1 =$$

$$-\frac{k_\gamma}{\mu_{const.}} \cdot \int\limits_0^{L(\alpha)} d\alpha_1 \int\limits_0^{L(\beta)} \left\{ \frac{p(\alpha,\beta,\gamma)\big|_{\gamma=L(\gamma)} - p(\alpha,\beta,\gamma)\big|_{\gamma=0}}{L(\gamma)} \right\} d\beta_1 =$$

$$= -\frac{k_\gamma}{\mu_{const.}} \cdot \frac{L(\alpha)\cdot L(\beta)}{L(\gamma)} \cdot \left\{ p(\alpha,\beta,\gamma)\big|_{\gamma=L(\gamma)} - p(\alpha,\beta,\gamma)\big|_{\gamma=0} \right\}. \tag{16}$$

As noted earlier, the porous truncated pyramid with permeability (1) under consideration has different filtration characteristics for the one-dimensional flows perpendicular to the level surfaces $\alpha(x,y,z)=\alpha_0 \equiv const.$, $\beta(x,y,z)=\beta_0 \equiv const.$, and $\gamma(x,y,z)=\gamma_0 \equiv const.$ Therefore, in the modelling process, it is necessary to replace the approximating porous curvilinear layered truncated pyramid with a "fictitious" anisotropic medium to preserve identical filtration characteristics for the same flows. Therefore, it is necessary to choose the corresponding parameters of the approximating "fictitious" anisotropic medium in such a way that the filtration properties of the modelled porous isotropic truncated pyramid, whose periodic permeability is given by the multiplicative formula (1), are identical for all three one-dimensional flows. For this purpose, we will match each of the formulas (9)-(11) with each of the corresponding formulas (14)-(16):

– from (9) and (14), we obtain that in the local Cartesian coordinate system $\alpha\times\beta\times\gamma$ within the small volume, the permeability k_α of the linear anisotropic porous medium along the local Cartesian axis α it is necessary to choose (adjust; determine) as:

$$k_\alpha = \frac{L(\beta)\cdot L(\gamma)}{L(\alpha)} \cdot \frac{\int\limits_0^{L(\beta)} K^{\{2\}}(\beta_1)d\beta_1 \cdot \int\limits_0^{L(\gamma)} K^{\{3\}}(\gamma_1)d\gamma_1}{\int\limits_0^{L(\alpha)} \frac{d\alpha_1}{K^{\{1\}}(\alpha_1)}}, \tag{17}$$

– from (10) and (15) we get that the permeability k_β of this medium needs to be chosen as:

$$k_\beta = \frac{L(\alpha)\cdot L(\gamma)}{L(\beta)} \cdot \frac{\int\limits_0^{L(\alpha)} K^{\{1\}}(\alpha_1)d\alpha_1 \cdot \int\limits_0^{L(\gamma)} K^{\{3\}}(\gamma_1)d\gamma_1}{\int\limits_0^{L(\beta)} \frac{d\beta_1}{K^{\{2\}}(\beta_1)}}, \tag{18}$$

– from (11) and (16) we get that the permeability k_γ of that porous medium needs to be chosen as:

$$k_\gamma = \frac{L(\alpha)\cdot L(\beta)}{L(\gamma)} \cdot \frac{\int\limits_0^{L(\alpha)} K^{\{1\}}(\alpha_1)\,d\alpha_1 \cdot \int\limits_0^{L(\beta)} K^{\{2\}}(\beta_1)\,d\beta_1}{\int\limits_0^{L(\gamma)} \frac{d\gamma_1}{K^{\{3\}}(\gamma_1)}}. \tag{19}$$

Let's note that the direct substitution of the expressions (12) and (13) into (9)-(11) for each elementary approximation-averaging parallelepiped gives us the following results:

$$Q_\alpha = -\frac{p(\alpha,\beta,\gamma)\big|_{\alpha=L(\alpha)} - p(\alpha,\beta,\gamma)\big|_{\alpha=0}}{\mu_{const.}\cdot \int\limits_0^{L(\alpha)} \frac{d\alpha_1}{k_\alpha}} \cdot \int\limits_0^{L(\beta)} k_\beta d\beta_1 \cdot \int\limits_0^{L(\gamma)} k_\gamma d\gamma_1 = \tag{20}$$
$$= -\frac{k_\alpha \cdot k_\beta \cdot k_\gamma}{\mu_{const.}} \cdot \frac{L(\beta)\cdot L(\gamma)}{L(\alpha)} \cdot \left\{ p(\alpha,\beta,\gamma)\big|_{\alpha=L(\alpha)} - p(\alpha,\beta,\gamma)\big|_{\alpha=0} \right\},$$

$$Q_\beta = -\frac{p(\alpha,\beta,\gamma)\big|_{\beta=L(\beta)} - p(\alpha,\beta,\gamma)\big|_{\beta=0}}{\mu_{const.}\cdot \int\limits_0^{L(\beta)} \frac{d\beta_1}{k_\beta}} \cdot \int\limits_0^{L(\alpha)} k_\alpha d\alpha_1 \cdot \int\limits_0^{L(\gamma)} k_\gamma d\gamma_1 = \tag{21}$$
$$= -\frac{k_\alpha \cdot k_\beta \cdot k_\gamma}{\mu_{const.}} \cdot \frac{L(\alpha)\cdot L(\gamma)}{L(\beta)} \cdot \left\{ p(\alpha,\beta,\gamma)\big|_{\beta=L(\beta)} - p(\alpha,\beta,\gamma)\big|_{\beta=0} \right\},$$

$$Q_\gamma = -\frac{p(\alpha,\beta,\gamma)\big|_{\gamma=L(\gamma)} - p(\alpha,\beta,\gamma)\big|_{\gamma=0}}{\mu_{const.}\cdot \int\limits_0^{L(\gamma)} \frac{d\gamma_1}{k_\gamma}} \cdot \int\limits_0^{L(\alpha)} k_\alpha d\alpha_1 \cdot \int\limits_0^{L(\beta)} k_\beta d\beta_1 = \tag{22}$$
$$= -\frac{k_\alpha \cdot k_\beta \cdot k_\gamma}{\mu_{const.}} \cdot \frac{L(\alpha)\cdot L(\beta)}{L(\gamma)} \cdot \left\{ p(\alpha,\beta,\gamma)\big|_{\gamma=L(\gamma)} - p(\alpha,\beta,\gamma)\big|_{\gamma=0} \right\}.$$

It is evident that the formulas (20)-(22) differ significantly from the formulas (14)-(16).

Similarly to the reasoning conducted before obtaining the formulas (17)-(19), it is necessary to compare each of the formulas (9)-(11) with each of the corresponding formulas (20)-(22). Such a comparison yields the following interesting results:

$$k_\alpha \cdot k_\beta \cdot k_\gamma = \frac{L(\alpha)}{L(\beta)\cdot L(\gamma)} \cdot \frac{\int\limits_0^{L(\beta)} K^{\{2\}}(\beta_1)\,d\beta_1 \cdot \int\limits_0^{L(\gamma)} K^{\{3\}}(\gamma_1)\,d\gamma_1}{\int\limits_0^{L(\alpha)} \frac{d\alpha_1}{K^{\{1\}}(\alpha_1)}} =$$
$$= \frac{L(\beta)}{L(\alpha)\cdot L(\gamma)} \cdot \frac{\int\limits_0^{L(\alpha)} K^{\{1\}}(\alpha_1)\,d\alpha_1 \cdot \int\limits_0^{L(\gamma)} K^{\{3\}}(\gamma_1)\,d\gamma_1}{\int\limits_0^{L(\beta)} \frac{d\beta_1}{K^{\{2\}}(\beta_1)}} =$$

$$=\frac{L(\gamma)}{L(\alpha)\cdot L(\beta)}\cdot\frac{\int\limits_0^{L(\alpha)}K^{\{1\}}(\alpha_1)d\alpha_1\cdot\int\limits_0^{L(\beta)}K^{\{2\}}(\beta_1)d\beta_1}{\int\limits_0^{L(\gamma)}\frac{d\gamma_1}{K^{\{3\}}(\gamma_1)}}.$$

Hence, it is easy to see that

$$\frac{1}{L^2(\alpha)}\cdot\int\limits_0^{L(\alpha)}K^{\{1\}}(\alpha_1)d\alpha_1\cdot\int\limits_0^{L(\alpha)}\frac{d\alpha_1}{K^{\{1\}}(\alpha_1)}=\frac{1}{L^2(\beta)}\cdot\int\limits_0^{L(\beta)}K^{\{2\}}(\beta_1)d\beta_1\cdot\int\limits_0^{L(\beta)}\frac{d\beta_1}{K^{\{2\}}(\beta_1)}=$$

$$=\frac{1}{L^2(\gamma)}\cdot\int\limits_0^{L(\gamma)}K^{\{3\}}(\gamma_1)d\gamma_1\cdot\int\limits_0^{L(\gamma)}\frac{d\gamma_1}{K^{\{3\}}(\gamma_1)}.$$

3. Mathematical model (inverse problem) for determining the periodic permeability coefficient of an isotropic medium (truncated pyramid) with a given period

Let the assumptions (2)-(4) are satisfied. Let us consider the equations of the filtration process:

– equation for the pressure $p(\alpha,\beta,\gamma)$

$$\frac{\partial}{\partial\alpha}\left\{K(\alpha,\beta,\gamma)\cdot\frac{\partial p(\alpha,\beta,\gamma)}{\partial\alpha}\right\}+\frac{\partial}{\partial\beta}\left\{K(\alpha,\beta,\gamma)\cdot\frac{\partial p(\alpha,\beta,\gamma)}{\partial\beta}\right\}+ +\frac{\partial}{\partial\gamma}\left\{K(\alpha,\beta,\gamma)\cdot\frac{\partial p(\alpha,\beta,\gamma)}{\partial\gamma}\right\}=0, \tag{23}$$

Where $(\alpha,\beta,\gamma)\in\Omega_{\alpha\times\beta\times\gamma}$; $\Omega_{\alpha\times\beta\times\gamma}\in R^3$ is an open elementary approximation-averaged parallelepiped in a curvilinear local Cartesian coordinate system $\alpha\times\beta\times\gamma$, moreover $\partial\Omega_{\alpha\times\beta\times\gamma}=\Gamma_\alpha\bigcup\Gamma_\beta\bigcup\Gamma_\gamma:\ \Gamma_\alpha\bigcap\Gamma_\beta=\varnothing,\ \Gamma_\alpha\bigcap\Gamma_\gamma=\varnothing,\ \Gamma_\beta\bigcap\Gamma_\gamma=\varnothing;$

– equation for the velocity field $\vec{\vartheta}(\alpha,\beta,\gamma)$

$$\vec{\vartheta}(\alpha,\beta,\gamma)=-\frac{K(\alpha,\beta,\gamma)}{\mu(\alpha,\beta,\gamma)}\cdot\nabla p(\alpha,\beta,\gamma), \tag{24}$$

Where $(\alpha,\beta,\gamma)\in\bar{\Omega}_{\alpha\times\beta\times\gamma}\overset{def}{\equiv}\Omega_{\alpha\times\beta\times\gamma}\bigcup\partial\Omega_{\alpha\times\beta\times\gamma}$; function $\mu(\alpha,\beta,\gamma)$ is a dynamic viscosity of the media.

In (23), (24) function $K(\alpha,\beta,\gamma)$, $(\alpha,\beta,\gamma)\in\bar{\Omega}_{\alpha\times\beta\times\gamma}$ is unknown permeability coefficient of the elementary approximation-averaged parallelepiped (in general, it is defined everywhere in the porous truncated pyramid) and it is assumed to be a periodic function with respect to all three curvilinear arguments. Moreover, it is represented as a multiplicative function, separated by curvilinear arguments (Aziz & Settari, 1982; Basniev *et al.*, 2006):

$$K(\alpha,\beta,\gamma)=K^{\{1\}}(\alpha)\cdot K^{\{2\}}(\beta)\cdot K^{\{3\}}(\gamma),\quad(\alpha,\beta,\gamma)\in\bar{\Omega}_{\alpha\times\beta\times\gamma}.$$

To equations (23), (24), let's add the following boundary conditions:

$$p(\alpha,\beta,\gamma)\big|_{(\alpha,\beta,\gamma)\in\Gamma_\alpha}=p_{\Gamma_\alpha}(\alpha,\beta,\gamma),\ (\alpha,\beta,\gamma)\in\Gamma_\alpha, \tag{25}$$

$$p(\alpha,\beta,\gamma)\big|_{(\alpha,\beta,\gamma)\in\Gamma_\beta} = p_{\Gamma_\beta}(\alpha,\beta,\gamma),\ \ (\alpha,\beta,\gamma)\in\Gamma_\beta, \tag{26}$$

$$\left\{K^{\{1\}}(\alpha)\cdot\frac{\partial p(\alpha,\beta,\gamma)}{\partial\alpha}\cdot n_\alpha + K^{\{2\}}(\beta)\cdot\frac{\partial p(\alpha,\beta,\gamma)}{\partial\beta}\cdot n_\beta + \right.$$
$$\left. + K^{\{3\}}(\gamma)\cdot\frac{\partial p(\alpha,\beta,\gamma)}{\partial\gamma}\cdot n\right\}\Bigg|_{(\alpha,\beta,\gamma)\in\Gamma_\gamma} = p_{\Gamma_\gamma}(\alpha,\beta,\gamma),\ \ (\alpha,\beta,\gamma)\in\Gamma_\gamma, \tag{27}$$

Where n_α, n_β and n_γ are direction cosines of the external normal $\vec{n}$.

In addition, it is assumed that the appropriate consistency conditions for the boundary functions $\bar{p}_{\Gamma_\alpha}(\alpha,\beta,\gamma)$, $\bar{p}_{\Gamma_\beta}(\alpha,\beta,\gamma)$ and $\bar{p}_{\Gamma_\gamma}(\alpha,\beta,\gamma)$ are satisfied.

It is required to determine the unknown functions $K^{\{i\}}(\omega_i)\in C^1(\bar{\Omega}_{\alpha\times\beta\times\gamma})$, $\{\omega_1;\omega_2;\omega_3\}=\{\alpha;\beta;\gamma\}$ $(i=\overline{1,3})$ in equations (23)-(27) under the following conditions:

– periods $T(\omega_i)$, $\{\omega_1;\omega_2;\omega_3\}=\{\alpha;\beta;\gamma\}$ $(i=\overline{1,3})$ of the functions $K^{\{i\}}(\omega_i)$, $\{\omega_1;\omega_2;\omega_3\}=\{\alpha;\beta;\gamma\}$ $(i=\overline{1,3})$ a priori known;

– functions $\mu(\alpha,\beta,\gamma)$, $(\alpha,\beta,\gamma)\in\bar{\Omega}_{\alpha\times\beta\times\gamma}$, $p_{\Gamma_\alpha}(\alpha,\beta,\gamma)$, $(\alpha,\beta,\gamma)\in\Gamma_\alpha$, $p_{\Gamma_\beta}(\alpha,\beta,\gamma)$, $(\alpha,\beta,\gamma)\in\Gamma_\beta$ and $p_{\Gamma_\gamma}(\alpha,\beta,\gamma)$, $(\alpha,\beta,\gamma)\in\Gamma_\gamma$ are given functions of their arguments;

– additional information is known a priori:

$$\frac{\partial p(\alpha,\beta,\gamma)}{\partial\alpha}\bigg|_{(\alpha,\beta,\gamma)\in\Gamma_\alpha} = \bar{p}_{\Gamma_\alpha}(\alpha,\beta,\gamma),\ \ (\alpha,\beta,\gamma)\in\Gamma_\alpha, \tag{28}$$

$$\frac{\partial p(\alpha,\beta,\gamma)}{\partial\beta}\bigg|_{(\alpha,\beta,\gamma)\in\Gamma_\beta} = \bar{p}_{\Gamma_\beta}(\alpha,\beta,\gamma),\ \ (\alpha,\beta,\gamma)\in\Gamma_\beta, \tag{29}$$

$$\frac{\partial p(\alpha,\beta,\gamma)}{\partial\gamma}\bigg|_{(\alpha,\beta,\gamma)\in\Gamma_\gamma} = \bar{p}_{\Gamma_\gamma}(\alpha,\beta,\gamma),\ \ (\alpha,\beta,\gamma)\in\Gamma_\gamma, \tag{30}$$

Where functions $\bar{p}_{\Gamma_\alpha}(\alpha,\beta,\gamma)$, $(\alpha,\beta,\gamma)\in\Gamma_\alpha$, $\bar{p}_{\Gamma_\beta}(\alpha,\beta,\gamma)$, $(\alpha,\beta,\gamma)\in\Gamma_\beta$ and $\bar{p}_{\Gamma_\gamma}(\alpha,\beta,\gamma)$, $(\alpha,\beta,\gamma)\in\Gamma_\gamma$ are also assumed to be known functions of their arguments, and all necessary consistency conditions are also satisfied for them.

The problem (23)-(30), together with all the necessary consistency conditions, constitutes a complete formulation of the inverse problem for determining the periodic permeability coefficient of an isotropic porous medium within each of its elementary approximating-averaging parallelepipeds.

4. Mathematical model (inverse problem) for determining the optimal spatial distribution of gas-bearing sources in the "pyramidal" problem

Let there be the region $(x,y)\in\partial\Omega,\ t\in(0,T]$, where $\Omega\in R^2$ is a two-dimensional filtration area of slightly compressible liquid (in our case, chemical composition) in a porous medium, $\partial\Omega=\bigcup_{i=1}^{K}G_i$ is the boundary of this area, moreover $G_i\bigcap G_j=\varnothing\ i\neq j\ \forall i,j=\overline{1,K},$ and T represents the time, during which the two-dimensional filtration process is studied (i.e. the planned period of the process; for instance, Aziz & Settari, 1982), satisfying:

– equation (Abdullaev, 1990)

$$Q(x,y)\cdot\left(\alpha\cdot\beta_{liquid}+\beta_{porous}\right)\cdot\frac{\partial p(x,y,t)}{\partial t}=$$
$$=div\left(\frac{k(x,y)\cdot Q(x,y)}{\mu(x,y)}\cdot\nabla p(x,y,t)\right)+\sum_{i=1}^{N}f^{\{i\}}(t)\cdot\delta\left(x-x^{\{i\}},y-y^{\{i\}}\right)=0, \tag{31}$$

– initial condition

$$p(x,y,t)\big|_{t=0}=p_0(x,y),\ (x,y)\in\Omega; \tag{32}$$

– boundary conditions

$$p(x,y,t)\big|_{(x,y)\in G_i}=p_1^{\{i\}}(t),\ i=\overline{1,K_1},\ t\in[0,T],\ \ K_1+K_2=K; \tag{33}$$

$$\left.\frac{\partial p(x,y,t)}{\partial\vec{n}}\right|_{(x,y)\in G_i}=p_2^{\{i\}}(t),\ i=\overline{1,K_2},\ t\in[0,T],\ \ K_1+K_2=K. \tag{34}$$

In (31)-(34), $p(x,y,t)$ denotes pressure at point $(x,y)\in\Omega$ at time t; and $k(x,y)$ denotes the permeability; $Q(x,y)$ – the power of the porous medium (indeed, the medium is divided into parallel layers (along the outer normal to the base of the pyramid) and function $Q(x,y)$ in this case is the power of the layer/stratum); $\mu(x,y)$ – viscosity of chemical composition; $\alpha=\alpha(x,y)$ – porosity; β_{liquid} – coefficient of compressibility of the fluid (chemical composition); β_{porous} – coefficient of porosity of the medium; N – number of sources; K – number of non-intersecting parts of the boundary of the two-dimensional filtration area; $\left(x^{\{i\}},y^{\{i\}}\right)$ are coordinates of the i-th $\left(i=\overline{1,N}\right)$ source location; $f^{\{i\}}(t)$ is the power of i-th $\left(i=\overline{1,N}\right)$ source (analogy of flow rate in underground hydrodynamics); $\delta\left(x-x^{\{i\}},y-y^{\{i\}}\right)$ is a two-dimensional Dirac delta function applied to the coordinates $\left(x^{\{i\}},y^{\{i\}}\right)$ of i-th $\left(i=\overline{1,N}\right)$ source; functions $p_0(x,y)$, $p_1^{\{i\}}(t)$ $\left(i=\overline{1,K_1}\right)$ and $p_2^{\{i\}}(t)$ $\left(i=\overline{1,K_2}\right)$ are given initial and boundary functions; $\vec{n}=\vec{n}(x,y)$ – external normal at point $(x,y)\in\Omega$.

It is required to determine from (31)-(34):

– Function $p(x,y,t)$ in the area $\Omega\times[0,T]$;

– Coordinates $\left(x^{\{i\}},y^{\{i\}}\right)$ $i=\overline{1,N}$ of all N sources (coordinates of some sources, for example, the first N_1 sources, may be known in advance. In this case, it is necessary to determine the coordinates of the remaining $N_2=N-N_1$ sources:

$\left(x^{\{i\}}, y^{\{i\}}\right)$ $i=\overline{N_1+1, N}$) from (31)-(34) with the following additional requirements of an economic and technological nature:

- optimality criterion (Abdullaev, 1990)

$$W(z, f; p) \overset{def}{\equiv} \iint_{\Omega} \left\{ p(z,t)\big|_{t=T} - \frac{1}{mes\Omega} \cdot \iint_{\Omega} p(\zeta,t)\big|_{t=T} d\zeta \right\}^2 dz + \\ +\omega_1 \cdot \sum_{i=1}^{N} \int_0^T \left\{ f^{\{i\}}(t) \right\}^2 dt + \omega_1 \cdot \sum_{i=1}^{N} \left\{ \left(x^{\{i\}}\right)^2 + \left(y^{\{i\}}\right)^2 \right\} \to \min, \tag{35}$$

- constraints

$$0 \le f_{\min}^{\{i\}} \le f^{\{i\}}(t) \le f_{\max}^{\{i\}} \quad \forall i = \overline{1, N}; \tag{36}$$

$$\sqrt{\left(x^{\{i\}} - x^{\{j\}}\right)^2 + \left(y^{\{i\}} - y^{\{j\}}\right)^2} \ge C_{\text{feasible distance}}^{\{i,j\}} \quad i \ne j \quad \forall i, j = \overline{1, N}; \tag{37}$$

$$\sum_{i=1}^{N} \int_0^T F\left(f^{\{i\}}(t) \right) dt \ge f_{\text{plan target}}. \tag{38}$$

In (35)-(38) $z = (x, y) \in \Omega$; $f = \left(f^{\{i\}}(t) \right)_{i=\overline{1,N}}$; $p_{\text{average}} \overset{def}{\equiv} \frac{1}{mes\Omega} \cdot \iint_{\Omega} p(\zeta,t)\big|_{t=T} d\zeta$ denotes the averaged value of the porous medium pressure; $\omega_1 > 0$ and $\omega_1 > 0$ are regularization parameters (criterion (35) is formulated taking into account Tikhonov's correctness for the optimization problem (35)-(38); Tikhonov & Arsenin, 1977); $f_{\min}^{\{i\}}$ denotes the minimum feasible power of the i-th $\left(i=\overline{1,N}\right)$ source; $f_{\max}^{\{i\}}$ denotes the maximum feasible power of the i-th $\left(i=\overline{1,N}\right)$ source; $C_{\text{feasible distance}}^{\{i,j\}}$ denotes the minimal feasible distance between the i-th and j-th sources $\left(i \ne j \;\&\; i, j=\overline{1,N}\right)$ (in the case of a uniform distribution of sources instead of $C_{\text{feasible distance}}^{\{i,j\}}$ we have the minimum feasible distance $C_{\text{feasible distance}}$, which is independent of the source numbers); $f_{\text{plan target}}$ is the plan target for the number of sources, which may be expressed in monetary units (in this case, the functional operator $F\left(f^{\{i\}}(t)\right)$ denotes the material benefits obtained from the operation of the i-th $\left(i=\overline{1,N}\right)$ source with power $f^{\{i\}}(t)$, or either expressed in volumes (in this case, the functional operator is the identity operator, i.e. $F\left(f^{\{i\}}(t)\right) \equiv f^{\{i\}}(t)$).

5. Conclusion

In this paper, the authors focus on an open landfill for solid consumer waste – a specially equipped area outdoors where municipal or private companies collect and dispose of organic and inorganic solid waste generated primarily from household consumption. From a mathematical perspective, the considered landfill for solid consumption waste represents a strongly isotropic layered porous medium, resembling a truncated pyramid in shape. The paper addresses three problems arising in the management of processes at open landfills for solid consumption waste to optimize biogas extraction processes. Mathematical models are developed for these problems, with one being a direct problem of mathematical physics and the other two being inverse problems: the coefficient inverse problem of mathematical physics and the inverse problem of nonlinear optimization. All three problems addressed in this paper have been solved using analytical and numerical methods and have been technically realized during the implementation of the scientific and innovative project 2010/0301/2DP/2.1.1.1.0/10/APIA/VIAA/151: "*New technology and software development for biogas extraction processes optimization*". More detailed information about this project is provided in the "Introduction" section of this paper. In this paper, the methods for solving these three models are not discussed.

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